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An extended Wirtinger inequality

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Abstract

Recently, an extended Wirtinger inequality proved extremely useful in studying the incipient relaxation dynamics of a nematic liquid crystal cell, in the presence of a weak anchoring potential. This inequality is proved here in detail and the specific dynamical problem to which it was first applied is also recalled.

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1. Introduction

In [1] we studied a mathematical model aiming at describing the behaviour of a novel liquid crystal display, originally proposed in [2], which appears to perform quickly enough to be compatible with video applications (see also [3] for a physical mechanism possibly explaining the fast switching involved in this device and [4] for a more mathematical account of it). An extended version of Wirtinger inequality in one dimension proved extremely useful in obtaining a characteristic time for the incipient dynamics of liquid crystals near a rigid material surface, which is crucial to estimate the applicability of our model to real devices. Here I present an elementary proof of this extended Wirtinger inequality, hoping that it would also be useful in the study of other mathematical models. The paper ends with a short account on the application of this inequality to the specific problem that prompted searching for it.

2. Inequality

Every function u of class C^1 on $[-h, h]$ such that

$$u(-h) = u(h) = 0$$

satisfies the following inequality:

$$\int_{-h}^{+h} u^2 \, dx \leqslant \left(\frac{2h}{\pi}\right)^2 \int_{-h}^{+h} u'^2 \, dx \tag{1}$$

where a prime denotes differentiation. This is the classical Wirtinger inequality (cf e.g. [5, p 185]), where the equality sign is attained if, and only if,

$$u(x) = c \cos \frac{\pi x}{2h}$$

for any constant c. Equivalently, inequality (1) can also be stated as follows:

$$\int_0^h u^2\,\mathrm{d}x \leqslant \left(\frac{2h}{\pi}\right)^2 \int_0^h u'^2\,\mathrm{d}x \tag{2}$$

provided that u obeys

$$u'(0) = 0 \qquad \text{and} \qquad u(h) = 0.$$

We prove here an inequality which reduces to (2) in a special case. Let $AC(0,h)$ denote the set of all absolutely continuous real-valued functions on the open interval $]0,h[$ and let $\mathcal{A}$ be the class of functions defined by

$$\mathcal{A} := \{u \in AC(0,h) : u'(0) = 0, u'(h)u(h) < 0\}.$$

The claim is that every $u \in \mathcal{A}$ satisfies the inequality

$$\int_0^h u'^2\,\mathrm{d}x - \frac{\gamma_u^2}{h^2}\int_0^h u^2\,\mathrm{d}x - u'(h)u(h) \geqslant 0 \tag{3}$$

where γ_u is the smallest root of the equation

$$-\gamma_u \tan\gamma_u = \frac{hu'(h)}{u(h)}. \tag{4}$$

3. Proof

To prove inequality (3) we first show that whenever $\gamma < \frac{\pi}{2}$ the functional

$$I_\gamma[u] := \int_0^h u'^2\,\mathrm{d}x - \frac{\gamma^2}{h^2}\int_0^h u^2\,\mathrm{d}x \tag{5}$$

attains its strict minimum in each class

$$\mathcal{A}_\alpha := \{u \in AC(0,h) : u'(0) = 0, u(h) = \alpha\}$$

for any given α, and that

$$\min_{\mathcal{A}_\alpha} I_\gamma = -\frac{\alpha^2}{h}\gamma\tan\gamma. \tag{6}$$

We preliminary remark that I_γ, for $\gamma < \frac{\pi}{2}$, is bounded from below in $\mathcal{A}_\alpha$ for all real α. In fact, by setting $u =: w + \alpha$, so that $w(h) = 0$, one readily arrives at

$$\begin{aligned} I_\gamma[u] &= \int_0^h w'^2\,\mathrm{d}x - \frac{\gamma^2}{h^2}\int_0^h (w^2 + \alpha^2 + 2\alpha w)\,\mathrm{d}x \\ &\geqslant \int_0^h w'^2\,\mathrm{d}x - \frac{\gamma^2}{h^2}(1 + \alpha\varepsilon^2)\int_0^h w^2\,\mathrm{d}x - \frac{\gamma^2\alpha}{h}\left(\alpha + \frac{1}{\varepsilon^2}\right) \end{aligned}$$

for every ε, where use has been made of the inequality

$$-2w \geqslant -\left(\frac{1}{\varepsilon^2} + \varepsilon^2 w^2\right).$$

Since $\gamma < \frac{\pi}{2}$, for any given α, ε can always be chosen so that

$$\gamma^2(1 + \alpha\varepsilon^2) < \frac{\pi^2}{4}.$$

Thus, by (2),

$$I_\gamma[u] \geqslant -\frac{\gamma^2\alpha}{2}\left(\alpha + \frac{1}{\varepsilon^2}\right).$$

The Euler–Lagrange equation associated with the functional I_γ is

$$u'' + \frac{\gamma^2}{h^2} u = 0$$

which is solved in $\mathcal{A}_\alpha$ by the function

$$u_0(x) = \frac{\alpha}{\cos\gamma} \cos\frac{\gamma x}{h}.$$

Moreover, by direct computation, one easily sees that the value of $I_\gamma[u_0]$ just equals the right-hand side of (6) for all values of γ. Since I_γ is a quadratic functional, it is proportional to its second variation:

$$\delta^2 I_\gamma(u)[v] = 2I_\gamma[v]$$

where v is any function in $AC(0, h)$ subject to

$$v'(0) = 0 \qquad \text{and} \qquad v(h) = 0.$$

Thus, by (2), $\delta^2 I_\gamma$ is positive definite, whenever $\gamma < \frac{\pi}{2}$, and so $I_\gamma[u_0]$ is the strict minimum of I_γ.

Let a function u be given in $\mathcal{A}$. By (4), this ensures that $\gamma_u < \frac{\pi}{2}$. Moreover, by setting $\alpha = u(h)$, we obtain from (6) that

$$\begin{aligned}\int_0^h u'^2\,\mathrm{d}x - \frac{\gamma_u^2}{h^2}\int_0^h u^2\,\mathrm{d}x - u'(h)u(h) &\geqslant \min_{\mathcal{A}_{u(h)}} I_{\gamma_u} - u'(h)u(h)\\ &= -\frac{u^2(h)}{h}\gamma_u \tan\gamma_u - u'(h)u(h)\end{aligned}$$

which by (4) yields (3). In the limit as $u_n(h) \to 0$ in a sequence u_n of functions in $\mathcal{A}$, $\gamma_{u_n} \to \frac{\pi}{2}$, and so inequality (3) reduces to (2).

4. Application

Here we apply inequality (3) to estimate the incipient growth of the solution to a specific partial differential equation, which arises in the relaxation dynamics of a liquid crystal cell, potentially of interest for the display industry (see [1] and [4]).

Let $(x, t) \mapsto \vartheta(x, t)$ be a function of $\boldsymbol{R}^+ \times \boldsymbol{R}^+$ into $[0, \frac{\pi}{2}]$ that solves the equation

$$\tau_s \vartheta_t = \xi_s^2 \vartheta_{xx} - d(x) \sin\vartheta \cos\vartheta \tag{7}$$

subject to

$$\vartheta_x|_{x=0} = 0 \qquad \lim_{x\to\infty} \vartheta_x = 0 \qquad \text{and} \qquad \vartheta|_{t=0} = \varphi(x)$$

where a subscript appended to ϑ denotes a partial derivative with respect to the corresponding variable, both τ_s and ξ_s are positive material constants, d is a positive function, decreasing to zero at infinity, and φ is a given function into $[0, \frac{\pi}{2}]$. The interested reader is referred to [1] for a derivation of equation (7); here we only recall that ϑ describes the orientation of the nematic director and that d represents the anchoring potential of a solid plate at $x = 0$, diluted over the region in space occupied by the liquid crystal, which decays considerably within a characteristic length h. Equation (7) combines together two distinctive features, each prevailing over the other either near the plate or away from it. For $x = 0$, at least as long as ϑ_{xx} does not grow too large, the evolution of ϑ is essentially driven by the relaxation term on the right-hand side of (7). In contrast, for $x \gg h$, where the anchoring potential has essentially faded away, the evolution of ϑ is just driven by diffusion.

We employ (3) to estimate precisely the early relaxation time associated with the given initial value φ. By setting $\vartheta =: \varphi + u$ and discarding terms of order two or higher in u, we obtain from (7) the following equation:

$$\tau_s u_t = \xi_s^2 u_{xx} - f(x)u + g(x) \tag{8}$$

subject to

$$u_x|_{x=0} = 0 \qquad \lim_{x\to\infty} u_x = 0 \qquad \text{and} \qquad u|_{t=0} \equiv 0$$

where

$$f := d\cos 2\varphi \qquad \text{and} \qquad g := \xi_s^2 \varphi_{xx} - \tfrac{1}{2} d \sin 2\varphi.$$

We assume that there is $h > 0$ and $T > 0$ such that the solution to (8) satisfies

$$u_x(h,t)u(h,t) < 0 \qquad \text{for all} \quad 0 < t < T$$

so that $u(\cdot, t) \in \mathcal{A}$ for all $0 < t < T$. Moreover, we define the following localized norm for u:

$$\|u\|_2 := \sqrt{\frac{1}{h}\int_0^h u^2\,\mathrm{d}x}.$$

By multiplying both sides of equation (8) by u and then integrating in x over the interval $[0, h]$, with the aid of (3) and the classical Cauchy–Schwarz inequality we arrive at

$$\tau_s \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 \leqslant -\frac{\xi_s^2\gamma_u^2}{h^2}\|u\|_2^2 + \|f\|_\infty \|u\|_2^2 + \|g\|_2\|u\|_2 \tag{9}$$

where $\|f\|_\infty$ is the supremum of f in $[0, h]$. Since $\|u\|_2$ vanishes at $t = 0$, as long as t is sufficiently small and to within terms smaller than $\|u\|_2^2$, in the right-hand side of (9) γ_u can be replaced by the root γ_0 of

$$-\gamma_0 \tan\gamma_0 = \lim_{t\to 0^+} \frac{hu_x|_{x=h}}{u|_{x=h}}.$$

Since

$$\int \frac{\mathrm{d}y}{ay + b\sqrt{y}} = \frac{2}{a}\ln(a\sqrt{y} + b)$$

for a and b real, integrating with respect to t in (9), we show that for t sufficiently small $\|u\|_2$ satisfies the following upper bound:

$$\|u\|_2 \leqslant \frac{\|g\|_2}{\frac{\xi_s^2\gamma_0^2}{h^2} - \|f\|_\infty}\left(1 - \exp\left[-\left(\frac{\xi_s^2\gamma_0^2}{h^2} - \|f\|_\infty\right)\frac{t}{\tau_s}\right]\right). \tag{10}$$

For a smooth solution of (7) such that $u_{xt}(h, 0) = u_{tx}(h, 0)$,

$$u(h,t) = u_t(h,0)t + \mathrm{o}(t) \qquad \text{and} \qquad u_x(h,t) = u_{tx}(h,0)t + \mathrm{o}(t)$$

and so, since $u|_{t=0} \equiv 0$, by (8) γ_0 can be estimated in terms of g as

$$-\gamma_0\tan\gamma_0 = \frac{hg'(h)}{g(h)}. \tag{11}$$

We call

$$\tau_g := \frac{\tau_s}{\left|\frac{\xi_s^2\gamma_0^2}{h^2} - \|f\|_\infty\right|}$$

the *incipient growth* time for the solution of (7). Depending on whether $\frac{\xi_s^2\gamma_0^2}{h^2}$ is larger or smaller than $\|f\|_\infty$, the incipient growth of ϑ is bounded or not.

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